SCHUR'S PARTITION THEOREM, COMPANIONS, REFINEMENTS AND GENERALIZATIONS

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ABSTRACT. Schur's partition theorem asserts the equality $S(n) = S_1(n)$, where $S(n)$ is the number of partitions of n into distinct parts $\equiv 1, 2 \pmod 3$ and $S_1(n)$ is the number of partitions of n into parts with minimal difference 3 and no consecutive multiples of 3. Using a computer search Andrews found a companion result $S(n) = S_2(n)$, where $S_2(n)$ is the number of partitions of n whose parts e_i satisfy $e_i - e_{i+1} \geq 3, 2$ or 5 according as $e_i \equiv 1, 2$ or $3 \pmod 3$. By means of a new technique called the method of weighted words, a combinatorial as well as a generating function proof of both these theorems are given simultaneously. It is shown that $S_1(n)$ and $S_2(n)$ are only two of six companion partition functions $S_j(n), j = 1, 2, ..., 6$, all equal to $S(n)$. A three parameter refinement and generalization of these results is obtained.

1. INTRODUCTION

The celebrated partition theorem which Schur [14] proved in 1926 is:

Theorem S. *Let $S(n)$ denote the number of partitions of n into distinct parts $\equiv 1, 2 \pmod 3$. Let $S_1(n)$ denote the number of partitions of n with minimal difference 3 between parts and such that no two consecutive multiples of 3 occur as parts. Then*

$$S(n) = S_1(n).$$

Several proofs of Schur's theorem and extensions of it are known. Andrews [5] gave a proof using generating functions and subsequently obtained extensions [6], [7] with 3 replaced by a larger modulus M and with gap conditions more complicated than those in Theorem S. In 1980, Bressoud [11] obtained a purely combinatorial proof and observed stronger correspondences between the partitions counted by $S(n)$ and $S_1(n)$.

In 1971 Andrews [8] found the following companion to Schur's theorem by a computer search:

Theorem A. *Let $S_2(n)$ denote the number of partitions of n in the form $n = e_1 + e_2 + \cdots + e_\nu$ such that $e_\ell - e_{\ell+1} \geq 3, 2$ or 5 if $e_\ell \equiv 1, 2$ or $3 \pmod 3$. Then*

$$S(n) = S_1(n) = S_2(n).$$

Andrews [8] gave a proof of Theorem A using generating functions in a manner similar to his previous proof of Theorem S. But the exact relationship between these two theorems remained unclear.

In a recent paper [1] we presented a new approach to Schur's theorem via what we called *the method of weighted words*. (See §2 for a description of this method and a summary of the results in [1].) This paper is a substantial extension of the ideas in [1] and has several important goals. Our first objective is to show that this method gives proofs of Theorems S and A simultaneously (see §3) and sets up a natural bijection between the partitions counted by $S_1(n)$ and $S_2(n)$. Our second, and more important, objective is to show that $S_1(n)$ and $S_2(n)$ are only two of six companion partition functions $S_\mu(n)$, $\mu = 1, 2, \ldots, 6$. (For the definitions of $S_3(n)$, $S_4(n)$, $S_5(n)$ and $S_6(n)$, see §4.) Our first main result is

Theorem 1. $S(n) = S_\mu(n)$, $\mu = 1, 2, \ldots, 6$.

The gap conditions defining the partition functions $S_\mu(n)$, $\mu = 3, 4, 5, 6$, are slightly more complicated and so did not show up in Andrews' computer search. For example, $S_3(n)$ is the number of partitions of n in the form $n = e_1 + e_2 + \cdots + e_\nu$ such that

$$(1.1) \qquad e_\ell - e_{\ell+1} \begin{cases} = 1 \text{ or } \geq 3, & \text{if } e_\ell \equiv 1 \pmod 3, \\ \geq 2 \text{ or } 6, & \text{if } e_\ell \equiv 2 \text{ or } 3 \pmod 3. \end{cases}$$

The method of weighted words provides a suitable general setting in which the relationship between the six partition functions can be seen clearly. In §5 we explain the six equalities in Theorem 1 by means of a *conservation law* for the gaps. Our method shows that the six companion functions $S_\mu(n)$ actually correspond to the six ways in which the q-multinomial coefficient

$$(1.2) \qquad \begin{bmatrix} i+j+k \\ i, j, k \end{bmatrix} = \frac{(q)_{i+j+k}}{(q)_i (q)_j (q)_k}$$

can be expanded (see §6).

We set up natural bijections between the partitions counted by each of the functions $S_\mu(n)$. As a consequence of these bijections several refinements of Theorem 1 emerge, and we state one now.

Theorem 2. *Let* $S_\mu(n; 3m)$ *denote the number of partitions counted by* $S_\mu(n)$ *with the added restriction that all parts are* $\leq 3m$. *Then for all integers* $n, m \geq 1$, *we have*

$$S_1(n; 3m) = S_2(n; 3m) = \cdots = S_6(n; 3m).$$

For a proof of Theorem 2 see §7. What is more, we show that the *a-b-c* generating function for these types of partitions counted by $S_\mu(n)$ are all equal, where the powers of the parameters a, b and c will represent the number of parts in residue classes 1, 2 and 3 (mod 3).

The third important objective in this paper is to show that the above theorems are special cases under the transformations

$$(1.3) \qquad \text{(dilation)} \quad q \mapsto q^3 \quad \text{and (translations)} \quad a \mapsto aq^{-2}, \quad b \mapsto bq^{-1}.$$

From now on we will refer to the transformations in (1.3) as *standard transformations*. More generally, under the dilation $q \mapsto q^M$ and translations

$a \mapsto aq^{\alpha-M}$, $b \mapsto bq^{\beta-M}$, one can get corresponding results with the modulus $M \geq 3$ replacing 3, where the powers of a and b keep track of parts in two arbitrary distinct residue classes α and β (mod M). Thus the method of weighted words is acting at the base level without any dilations.

Recently, in collaboration with George Andrews, we have been able to extend the method of weighted words to obtain a new proof of deep partition theorems of Göllnitz ([13], Sätze 4.8 and 4.10) and to prove and refine a partition conjecture of Capparelli [12]. These will be discussed in forthcoming papers [2], [3]. Some remarks about these problems are made in § 8.

We use the standard notation

$$(a)_n = (a\,;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

for an integer $n \geq 1$, and

$$(a)_\infty = \lim_{n\to\infty}(a)_n\,, \quad \text{if } |q| < 1.$$

As usual, null products have value 1. The q-binomial coefficients are defined by

$$\begin{bmatrix} i+j \\ i,\,j \end{bmatrix} = \frac{(q)_{i+j}}{(q)_i(q)_j},$$

and the multinomial coefficients are given in (1.2).

Given a partition π, we let $\sigma(\pi)$ denote the sum of its parts, $\lambda(\pi)$ its largest part and $\nu(\pi)$ the number of parts of π. Quite often we will be counting the number of parts of π of a specific type, and this will be indicated by a subscript, eg: $\nu_a(\pi)$ will be the number of a-parts of π (precise definitions will be given when necessary). Sometimes the parts will be arranged according to a specific lexicographic ordering (not necessarily the standard ordering of the positive integers). In such cases $\lambda(\pi)$ will denote the largest part of π with respect to this ordering. We also adopt the convention that $j \equiv b$ (mod m) means that $j = b + \ell m$ with $j > 0$ and $\ell \geq 0$.

Schur had originally stated the equality of three partition functions, $T(n) = S(n) = S_1(n)$, where $T(n)$ is number of partitions of n into parts $\equiv \pm 1$ (mod 6), repetitions allowed. However, in discussing $T(n)$, one does not get refinements; that is, it is not possible to keep track of the number of parts in specific residue classes as can be done with the partitions counted by $S(n)$ and $S_1(n)$. Hence we do not discuss the function $T(n)$ in this paper.

2. THE METHOD OF WEIGHTED WORDS

Let each integer $j \geq 2$ occur in three colors, red, blue and purple, and let the integer 1 occur only in two colors, red and blue. We introduce the symbols a_j, b_j and c_j to represent the integer (part) j in colors red, blue and purple respectively, and sometimes we simply refer to a_j as an a-part, b_j as a b-part and c_j as a c-part. We think of the letters a, b and c as representing free parameters. Under the standard transformations, the powers of the parameters a, b and c will represent the number of parts congruent to 1, 2 and 3 (mod 3) respectively in partitions counted by $S_\mu(n)$. However, in making the transition from $S_\mu(n)$ to $S(n)$, we will need to choose $c = ab$. That is why we think of c having color purple = red plus blue.

The *gap* between any two symbols representing colored integers is defined to be the absolute value of the difference between their subscripts. The gap

therefore is a non-negative integer, without color. For example, the gap between a_5 and c_5 is 0, and between a_5 and b_7 is 2. The weight of a symbol is defined to be its subscript. For example, the weight of b_6 is 6, and b_6 represents the integer 6 colored blue. Now in order to discuss partitions using these symbols, we need a lexicographic ordering of the symbols, and that is what we discuss next.

First we consider the following lexicographic ordering:

$$(2.1) \qquad a_1 \prec b_1 \prec c_2 \prec a_2 \prec b_2 \prec c_3 \prec a_3 \prec b_3 \prec \cdots .$$

We shall refer to this as Scheme 1. Now by a partition π, we mean a sum involving the symbols a_j, b_j and c_j where the symbols (parts) are arranged in decreasing order as given by Scheme 1. Instead of thinking of π as a sum we may think of it as a word with symbols in decreasing order. By $\sigma(\pi) = n$ we mean that the sum of the weights of the symbols of π is n. So π is a partition of n into colored integers. For example $\pi = a_7 + c_7 + b_5 + b_5 + a_4 + c_4 + c_2 + b_1 + a_1 + a_1$ is a partition of 37 because $\sigma(\pi) = 37$. We let $\nu_a(\pi)$, $\nu_b(\pi)$ and $\nu_c(\pi)$ denote the number of a-parts, b-parts and c-parts of π. In this example $\nu_a(\pi) = 4$, $\nu_b(\pi) = 3$ and $\nu_c(\pi) = 3$. In this example the largest part $\lambda(\pi) = a_7$.

Next we consider a partition $e_1 + e_2 + \cdots + e_\nu$ using symbols from Scheme 1 such that the gap between the symbols is ≥ 1, with the added restriction that the gap between consecutive symbols e_ℓ, $e_{\ell+1}$ is > 1 if

$$(2.2) \qquad \left. \begin{array}{l} e_\ell \text{ is red, } e_{\ell+1} \text{ is blue} \\ \text{or } e_\ell \text{ is purple.} \end{array} \right\}$$

We will refer to this as a Type 1 partition. The following result was established in [1]:

Lemma 1. *Let* $G = G(i, j, k)$ *be the generating function for all Type* 1 *partitions* π *having* $\nu_a(\pi) = i$, $\nu_b(\pi) = j$ *and* $\nu_c(\pi) = k$. *That is,*

$$G = \sum_{\substack{\pi \text{ of Type } 1 \\ \nu_a(\pi)=i,\ \nu_b(\pi)=j,\ \nu_c(\pi)=k}} q^{\sigma(\pi)} .$$

Then

$$G = \frac{q^{T_{i+j+k}+T_k}}{(q)_i (q)_j (q)_k} ,$$

where $T_m = m(m+1)/2$ *is the* mth *triangular number.*

We describe briefly the ideas behind the proof of Lemma 1, as this will be useful for more general results in the following sections.

Consider the generating function for all Type 1 partitions where the numbers i, j, k of colored parts are specified and the order in which they occur is also given. Then the generating function can be computed as follows: First construct the minimal partition π satisfying the restrictions. Then the required generating function is

$$(2.3) \qquad q^{\sigma(\pi)} / (q)_{\nu(\pi)}.$$

Since Type 1 partitions are given by lower bound gap conditions, the partitions in question are obtained by adding to π all (ordinary) partitions ρ with $\nu(\rho) \leq$

$\nu(\pi)$. Thus the expression in (2.3) is the required generating function. For example, if the colors occur in the order

$$c \quad a \quad b \quad b \quad c,$$

then the minimal Type 1 partition is

$$\pi = c_8 + a_6 + b_4 + b_3 + c_2.$$

So $\sigma(\pi) = 23$, $\nu(\pi) = 5$, and the required generating function is

$$q^{23}/(q)_5.$$

In view of these observations, write the expression for G in Lemma 1 as

$$(2.4) \qquad G = \frac{q^{T_{i+j+k}+T_k}}{(q)_{i+j+k}} \begin{bmatrix} i+j+k \\ i,j,k \end{bmatrix}.$$

Since $\nu(\pi) = i + j + k$ for all partitions counted by G, we see from (2.3) and (2.4) that Lemma 1 is equivalent to the statement that the generating function $H = H(i, j, k)$ for all minimal partitions having i red, j blue and k purple parts is

$$(2.5) \qquad H(i, j, k) = q^{T_{i+j+k}+T_k} \begin{bmatrix} i+j+k \\ i,j,k \end{bmatrix}.$$

To prove (2.5) consider the sets A, B and C of all partitions of Type 1 such that the smallest part is red, blue and purple respectively. Then consider the generating functions

$$(2.6) \qquad H_a(i, j, k) = \sum_{\mu \in A} q^{\sigma(\mu)}, \qquad H_b(i, j, k) = \sum_{\mu \in B} q^{\sigma(\mu)},$$

$$H_c(i, j, k) = \sum_{\mu \in C} q^{\sigma(\mu)}.$$

So

$$(2.7) \qquad H = H_a + H_b + H_c.$$

It can be shown by induction on $i + j + k$ (see [1]) that

$$H_a(i, j, k) = q^{T_{i+j+k}+T_k} \begin{bmatrix} i+j+k-1 \\ i-1,j,k \end{bmatrix},$$

$$(2.8) \qquad H_b(i, j, k) = q^{T_{i+j+k}+T_k+i} \begin{bmatrix} i+j+k-1 \\ i,j-1,k \end{bmatrix},$$

$$H_c(i, j, k) = q^{T_{i+j+k}+T_k+i+j} \begin{bmatrix} i+j+k-1 \\ i,j,k-1 \end{bmatrix}.$$

So (2.5) follows from (2.7) and (2.8) because of the recurrence
(2.9)
$$\begin{bmatrix} i+j+k \\ i,j,k \end{bmatrix} = \begin{bmatrix} i+j+k-1 \\ i-1,j,k \end{bmatrix} + q^i \begin{bmatrix} i+j+k-1 \\ i,j-1,k \end{bmatrix} + q^{i+j} \begin{bmatrix} i+j+k-1 \\ i,j,k-1 \end{bmatrix}$$

satisfied by the q-multinomial coefficients.

Next, let $B_1(n; i, j, k)$ denote the number of Type 1 partitions π of n having $\nu_a(\pi) = i$, $\nu_b(\pi) = j$, and $\nu_c(\pi) = k$. Then from Lemma 1 we see that

(2.10)
$$\sum_{i,j,k,n} B(n; i, j, k)a^i b^j c^k q^n = \sum_{i,j,k} a^i b^j c^k G(i, j, k)$$
$$= \sum_{i,j,k} a^i b^j c^k \frac{q^{T_{i+j+k}+T_k}}{(q)_i (q)_j (q)_k}.$$

Our next result is

Lemma 2. *Let* $r, s \geq 0$ *be given integers. Then*

$$\sum_{0 \leq m \leq \min(r,s)} G(r-m, s-m, m) = \sum_{0 \leq m \leq \min(r,s)} \frac{q^{T_{r+s-m}+T_m}}{(q)_{r-m}(q)_{s-m}(q)_m} = \frac{q^{T_r+T_s}}{(q)_r(q)_s}.$$

In [1] we give two proofs of Lemma 2, one of which is combinatorial. Lemma 2 has a nice combinatorial interpretation from which a refinement of Schur's theorem follows and we describe this next.

Note that

(2.11)
$$\sum_{r,s} \frac{a^r b^s q^{T_r+T_s}}{(q)_r(q)_s} = (-aq)_\infty (-bq)_\infty = \sum_{n,r,s} V(n; r, s)a^r b^s q^n ,$$

where $V(n; r, s)$ is the number of (vector) bi-partitions $(\pi_1; \pi_2)$ of n such that π_1 has r distinct red parts and π_2 has s distinct blue parts. So Lemma 2 is the assertion that the generating functions in (2.10) and (2.11) are equal when $c = ab$. In this case with

$$i = r - m, \qquad j = s - m, \qquad k = m$$

we have

(2.12)
$$i + j + 2k = r + s.$$

So we get the following combinatorial result:

Theorem 3. *Let* $t \geq 0$ *be given and* $V(n; t) = \sum_{r+s=t} V(n; r, s)$. *Then*

$$V(n; t) = \sum_{i+j+2k=t} B(n; i, j, k).$$

Theorem 3 gives a refinement of Theorem S under the standard transformations. More generally, under the dilation $q \mapsto q^M$, $a \mapsto aq^{\alpha-M}$, $b \mapsto bq^{\beta-M}$ applied to (2.10) and (2.11) we get the following refinement and generalization of Schur's theorem:

Theorem 4. *Let* $M \geq 3$ *and* $0 < \alpha < \beta < M \leq \alpha + \beta$.

Let $A(n; k)$ *denote the number of partitions of* n *into* k *distinct parts congruent to* α *or* β (mod M).

Let $B(n; k)$ *denote the number of partitions of* n *into* k *parts* $\equiv \alpha$, β *or* $\alpha + \beta$ (mod M) *such that*

(i) *the difference between any two parts is* $\geq M$,
(ii) *the difference between parts* $\equiv (\alpha + \beta)$ (mod M) *is* $> M$,
(iii) *the parts* $\equiv \alpha + \beta$ (mod M) *are counted twice.*

Then
$$A(n; k) = B(n; k).$$

In [1] a more general form of Theorem 4 was stated by relaxing the condition $0 < \alpha < \beta < M \leq \alpha + \beta$ to the simpler condition that α, β and $\alpha + \beta$ are incongruent (mod M). The reason that condition (2.12) enters into Theorem 3 is because, given a bi-partition of n into r distinct red parts and s distinct blue parts, one takes m of the red parts and m of the blue parts to form m purple parts, leaving behind $r - m$ red and $s - m$ blue parts. The combinatorial proof of Theorem S given by Bressoud [11] gives a different bijection from the one we provide for Theorem 3 in [1]. In the next section we describe our combinatorial proof of Theorem 3, and by changing the ordering in the penultimate step of this proof, we can get Theorem A instead of Theorem S.

3. COMBINATORIAL PROOF OF THEOREMS S AND A

Under the standard transformation the symbols become

$$(3.1) \qquad a_m = 3m - 2, \quad b_m = 3m - 1, \quad c_{m+1} = (ab)_{m+1} = 3m,$$

when $c = ab$. In this case the lexicographic ordering (2.1) of Scheme 1 yields the natural ordering
$$1 < 2 < 3 < 4 < \cdots$$
of the positive integers.

Let $x_m = x_m^{(1)}$ denote the symbol occupying the mth position in (2.1). That is, $x_1 = a_1$, $x_2 = b_1$, $x_3 = c_2$, $\cdots$, and so on. Instead of defining Type 1 partitions by means of the gap conditions (2.2), we may define them equivalently as follows: Type 1 partitions are those of the form $x_{m_1} + x_{m_2} + \cdots + x_{m_\nu}$, where

$$(3.2) \qquad m_\ell - m_{\ell+1} \geq 3, \quad \text{with strict inequality if } x_{m_\ell} \text{ is a } c\text{-part.}$$

This is a direct translation of the difference conditions defining $S_1(n)$ to the more general situation involving weighted symbols. From now on we will refer to the inequalities in (3.2) as *standard gap conditions*.

In order to understand the difference conditions defining Andrews' function $S_2(n)$, consider another lexicographic ordering of the symbols, namely, Scheme 2, given by

$$(3.3) \qquad a_1 \prec c_2 \prec b_1 \prec a_2 \prec c_3 \prec b_2 \prec a_3 \prec c_4 \prec b_3 \prec \cdots .$$

Under the standard transformations and with $c = ab$, Scheme 2 in (3.3) yields the following different ordering of the positive integers:

$$(3.4) \qquad 1 \prec 3 \prec 2 \prec 4 \prec 6 \prec 5 \prec 7 \prec 9 \prec 8 \prec \cdots$$

Now let $x_m^{(2)}$ denote the symbol occupying the mth position in (3.4). Then the difference conditions defining Andrews' function $S_2(n)$ are equivalent to the statement that we consider partitions of the form $x_{m_1}^{(2)} + x_{m_2}^{(2)} + \cdots + x_{m_\nu}^{(2)}$, where the $x_{m_\ell} = x_{m_\ell}^{(2)}$ satisfy the standard gap conditions (3.2). More generally, if $x_m^{(2)}$ denotes the symbol occupying the mth position in (3.3), let a Type 2 partition be one of the form $x_{m_1}^{(2)} + \cdots + x_{m_\nu}^{(2)}$, with $x_{m_\ell} = x_{m_\ell}^{(2)}$ satisfying (3.2). Also let $B_2(n; i, j, k)$ denote the number of Type 2 partitions π of n having $\nu_a(\pi) = i$, $\nu_b(\pi) = j$ and $\nu_c(\pi) = k$. We then have

Theorem 5. *Let* $n > 0$ *and* $i, j, k \geq 0$ *be integers. Then*

$$B_1(n; i, j, k) = B_2(n; i, j, k).$$

Theorem A is a consequence of Theorem 5 under the standard transformations.

We now discuss a combinatorial method which gives a bijective proof of Theorem A and more. The proof given here is an extension of the method in [1], which itself was based on earlier ideas of Bressoud [10].

Let r, s be given. We start with a bipartition $(\pi_1; \pi_2)$ counted by $V(n; r, s)$. So $\nu(\pi_1) = r$, $\nu(\pi_2) = s$. In what follows, several steps will be given and illustrated with

$$\pi_1 = a_7 + a_6 + a_3 + a_2 + a_1 \quad \text{and} \quad \pi_2 = b_{13} + b_{12} + b_8 + b_7 + b_5 + b_4 + b_2.$$

Step 1. Decompose π_2 into $\pi_4 + \pi_5$, where π_4 has the parts of π_2 which are $\leq r$ and π_5 has the remaining parts. Say, $\nu(\pi_4) = m$. So $\nu(\pi_5) = s - m$, and

$$\pi_4 = b_5 + b_4 + b_2, \qquad \pi_5 = b_{13} + b_{12} + b_8 + b_7.$$

Step 2. Consider the conjugate of the Ferrer's graph of π_4 and circle the bottom node in each column. Denote this graph by π_4^*. Construct the Ferrer's graph π_6, where the number of nodes in each row of π_6 is the sum of the number of nodes in the corresponding rows of π_1 and π_4^*. The parts of π_6 ending in the circled nodes are colored purple, and the remaining parts of π_6 are red. Thus π_6 has $r - m$ red parts and m purple parts, and

$$\pi_6 = \pi_1 + \pi_4^* = a_{10} + c_9 + a_5 + c_4 + c_2.$$

Step 3. Write the parts of π_5 in a column in descending order, and below them write the parts of π_6 in descending order. Draw a line to separate the parts of π_5 and π_6.

Step 4. Subtract 0 from the bottom element, 1 from the next element above that, 2 from the one above that, $\cdots$, and display the new values and the subtracted values in two adjacent columns $C_1 | C_2$. The elements of C_2 have no color, while those of C_1 retain the color of the parts from which they were derived.

Step 3	Step 4	
π_5/π_6	C_1	C_2
b_{13}	b_5	8
b_{12}	b_5	7
b_8	b_2	6
$\underline{b_7}$	$\underline{b_2}$	5
a_{10}	a_6	4
c_9	c_6	3
a_5	a_3	2
c_4	c_3	1
c_2	c_2	0

Step 5 (Penultimate Step). Rearrange the entries of C_1 in descending order according to Scheme 1 or Scheme 2, to form a column C_1^R.

Step 6 (Final Step). Add the corresponding elements of C_1^R and C_2 to get a partition counted by $B_1(n; r - m, s - m, m)$ or $B_2(n; r - m, s - m, m)$ depending upon which ordering was chosen at the penultimate step.

	Scheme 1				Scheme 2	
Step 5		Step 6		Step 5		Step 6
C_1^R	C_2	π_3		C_1^R	C_2	π_3
a_6	8	a_{14}		a_6	8	a_{14}
c_6	7	c_{13}		b_5	7	b_{12}
b_5	6	b_{11}		b_5	6	b_{11}
b_5	5	b_{10}		c_6	5	c_{11}
a_3	4	a_7		a_3	4	a_7
c_3	3	c_6		b_2	3	b_5
b_2	2	b_4		b_2	2	b_4
b_2	1	b_3		c_3	1	c_4
c_2	0	c_2		c_2	0	c_2

Every one of these steps is a one-to-one correspondence. So this procedure shows two things:

(i) Let r, s be given. Then

$$(3.4) \quad V(n; r, s) = \sum_m B_1(n; r - m, s - m, m) = \sum_m B_2(n; r - m, s - m, m),$$

(ii)

$$(3.5) \quad B_1(n; i, j, k) = B_2(n; i, j, k),$$
$$\text{taking } r - m = i, s - m = j, m = k.$$

Theorem 5 is a consequence of (3.5). Theorems S and A are consequences of (3.4) under the standard transformations.

To simply get a bijection between the partitions counted by $B_1(n; i, j, k)$ and $B_2(n; i, j, k)$, proceed from Step 6 to Step 5, replace the Scheme 1 ordering in C_1^R by the Scheme 2 ordering, and return to Step 6.

4. SIX SCHEMES AND SIX COMPANIONS TO SCHUR

Scheme 1 was generated by the standard ordering $a_1 \prec b_1 \prec c_2$ while Scheme 2 was generated by a different ordering $a_1 \prec c_2 \prec b_1$. More generally, we see that there are six schemes given by the six permutations of the symbols a_1, b_1

and c_2. They are

(4.1)   <u>Scheme 1</u>: $a_1 \prec b_1 \prec c_2 \prec a_2 \prec b_2 \prec c_3 \prec \cdots$

(4.2)   <u>Scheme 2</u>: $a_1 \prec c_2 \prec b_1 \prec a_2 \prec c_3 \prec b_2 \prec \cdots$

(4.3)   <u>Scheme 3</u>: $c_2 \prec a_1 \prec b_1 \prec c_3 \prec a_2 \prec b_2 \prec \cdots$

(4.4)   <u>Scheme 4</u>: $b_1 \prec a_1 \prec c_2 \prec b_2 \prec a_2 \prec c_3 \prec \cdots$

(4.5)   <u>Scheme 5</u>: $b_1 \prec c_2 \prec a_1 \prec b_2 \prec c_3 \prec a_2 \prec \cdots$

(4.6)   <u>Scheme 6</u>: $c_2 \prec b_1 \prec a_1 \prec c_3 \prec b_2 \prec a_2 \prec \cdots$

Actually, only three of the six schemes are essentially different, because Schemes 4, 5 and 6, are obtained from Schemes 1, 2 and 3 by interchanging the roles of a and b. But there are certain advantages in discussing all six schemes, as will be seen in the sequel.

Next, let $x_m^{(\mu)}$ denote the symbol occupying position m in Scheme μ, $\mu = 1, 2, \cdots, 6$. By a partition π of Type μ we mean an expression $x_{m_1}^{(\mu)} + x_{m_2}^{(\mu)} + \cdots + x_{m_t}^{(\mu)}$, where the $x_{m_t} = x_{m_t}^{(\mu)}$ satisfy the standard gap conditions (3.2). Let $B_\mu(n; i, j, k)$ denote the number of partitions π of n of Type μ such that $\nu_a(\pi) = i$, $\nu_b(\pi) = j$ and $\nu_c(\pi) = k$. Then the combinatorial method of §3 shows that

Theorem 6. *Let* $n \geq 1$ *and* $i, j, k \geq 0$ *be given integers. Then*

$$B_1(n; i, j, k) = B_2(n; i, j, k) = \cdots = B_6(n; i, j, k).$$

Theorem 7. *Let* $r, s \geq 0$ *be given integers. Then for* $\mu = 1, 2, \cdots, 6$, *we have*

$$V(n; r, s) = \sum_{0 \leq m \leq \min(r, s)} B_\mu(n; r - m, s - m, m).$$

Consequently, if $r + s = t$, *then*

$$V(n; t) = \sum_{i+j+2k=t} B_\mu(n; i, j, k).$$

Theorem 7 is a consequence of Theorem 6 and (3.4). To see the equality of the partition functions $B_\mu(n; i, j, k)$ and $B_\omega(n; i, j, k)$ one may use the ideas of §3 to construct a bijection $\mathscr{F}_{\mu, \omega}$ between the partitions π_μ counted by B_μ and the partitions π_ω counted by B_ω. Indeed, if π_μ is given, then write the parts of π_μ in a column in descending order as in Step 6. Now subtract $0, 1, 2, \cdots$, in succession from the bottom part upwards to get the columns $C_1^R | C_2$ of Step 5. The entries of C_1^R will be in descending order according to Scheme μ. Rearrange the entries of C_1^R according to the order given by Scheme ω and add back on the corresponding elements of C_2. Now we are back in Step 6 with a partition π_ω counted by B_ω. Thus

$$\mathscr{F}_{\mu, \omega}(\pi_\mu) = \pi_\omega$$

and the construction is reversible. That is,

$$\mathscr{F}_{\mu, \omega}^{-1} = \mathscr{F}_{\omega, \mu},$$

establishing the equality claimed in Theorem 6.

Under the standard transformations, the gap conditions (3.2) defining the Type μ partitions, $\mu = 1, 2, \cdots, 6$, become the following: We are now counting partitions of the form $e_1 + e_2 + \cdots + e_\nu$, where the e_ℓ are (ordinary) positive integers satisfying the difference conditions given by

(4.7)   <u>Type 1</u>: $e_\ell - e_{\ell+1} \geq 3, 3,$ or $4,$ if $e_\ell \equiv 1, 2,$ or 3 (mod 3).

(4.8)   <u>Type 2</u>: $e_\ell - e_{\ell+1} \geq 3, 2,$ or $5,$ if $e_\ell \equiv 1, 2,$ or 3 (mod 3).

(4.9)   <u>Type 3</u>: $\left. \begin{array}{l} e_\ell - e_{\ell+1} = 1 \text{ or } \geq 3, \quad \text{if } e_\ell \equiv 1 \text{ (mod 3)} \\ \qquad\qquad\quad\ \geq 2 \text{ or } 6, \quad \text{if } e_\ell \equiv 2 \text{ or } 3 \text{ (mod 3)} \end{array} \right\}.$

(4.10)   <u>Type 4</u>: $\left. \begin{array}{l} e_\ell - e_{\ell+1} \geq 2 \text{ or } 4, \quad \text{if } e_\ell \equiv 1 \text{ or } 3 \text{ (mod 3)} \\ \qquad\qquad\quad\ = 3 \text{ or } \geq 5, \quad \text{if } e_\ell \equiv 2 \text{ (mod 3)} \end{array} \right\}.$

(4.11)   <u>Type 5</u>: $\left. \begin{array}{l} e_\ell - e_{\ell+1} \geq 1, \quad \text{if } e_\ell \equiv 1 \text{ (mod 3)} \\ \qquad\qquad\quad\ = 3 \text{ or } \geq 5, \quad \text{if } e_\ell \equiv 2 \text{ (mod 3)} \\ \qquad\qquad\quad\ = 4 \text{ or } \geq 6, \quad \text{if } e_\ell \equiv 3 \text{ (mod 3)} \end{array} \right\}.$

(4.12)   <u>Type 6</u>: $\left. \begin{array}{l} e_\ell - e_{\ell+1} \geq 1 \text{ or } 6, \quad \text{if } e_\ell \equiv 1 \text{ or } 3 \text{ (mod 3)} \\ \qquad\qquad\quad\ = 2, 3 \text{ or } \geq 5, \quad \text{if } e_\ell \equiv 2 \text{ or } 3 \text{ (mod 3)} \end{array} \right\}.$

Note that the difference conditions in (4.7) are precisely those defining $S_1(n)$ in Theorem S and the conditions in (4.8) are the same as those given by Andrews for $S_2(n)$ in Theorem A.

Next let $S_\mu(n; i, j, k)$ denote the number of Type μ partitions of n having i parts $\equiv 1$ (mod 3), j parts $\equiv 2$ (mod 3) and k parts $\equiv 3$ (mod 3). Then using Theorems 6 and 7 and the standard transformations, Theorems S, A and 1 in § 1 can be improved to:

Theorem 8. *Given integers $n \geq 1$ and $i, j, k \geq 0$,*

$$S_1(n; i, j, k) = S_2(n; i, j, k) = \cdots = S_6(n; i, j, k).$$

Theorem 9. *Given integers $n \geq 1$ and $r, s \geq 0$, let $S(n; r, s)$ denote the number of partitions of n into r distinct parts $\equiv 1$ (mod 3) and s distinct parts $\equiv 2$ (mod 3). Then*

$$S(n; r, s) = \sum_{0 \leq m \leq \min(r,s)} S_\mu(n; r - m, s - m, m), \qquad \mu = 1, 2, \cdots, 6.$$

5. A CONSERVATION LAW FOR GAPS

The gap conditions in (2.2) which define the Type 1 partitions can be put in the following matrix form:

<u>Type 1</u>:

	a	b	c
a	1	1	2
b	2	1	2
c	1	1	2

The entries in this matrix are the minimum gaps allowed in Type 1 partitions. To be more precise, the entries $2, 1, 2$ in the second row have the following interpretation. If the weight of a certain b part is w and if the next bigger part is an a-part, then that a-part has weight $\geq w + 2$. If the next bigger part is a b-part, its weight is $\geq w + 1$, and if it is a c-part, its weight is $\geq w + 2$.

For Type 2 partitions, the gap conditions defining them can be embodied in the matrix

Type 2:

	a	b	c
a	1	1	2
b	2	1	3
c	1	0	2

There are two similarities between these two matrices. To describe this let x and y take values a, b and c and let $\gamma_1(x, y)$ (resp. $\gamma_2(x, y)$) denote the entry corresponding to row x and column y in the matrix of Type 1 (resp. Type 2). (For example $\gamma_1(b, a) = 2$ and $\gamma_2(b, c) = 3$.) Then the diagonal entries of the two matrices are the same, namely

(5.1)
$$\gamma_1(a, a) = \gamma_2(a, a) = 1, \quad \gamma_1(b, b) = \gamma_2(b, b) = 1, \quad \gamma_3(c, c) = \gamma_2(c, c) = 1.$$

The second similarity is that

(5.2)
$$\gamma_\mu(x, y) + \gamma_\mu(y, x) = 3, \quad \mu = 1, 2, \ x \neq y.$$

In other words, the conservation law is that the sum of the gaps between x, y and y, x is ≥ 3.

The matrices defining the other four types of partitions are given by

Type 3:

	a	b	c
a	1	1	3
b	2	1	3
c	0	0	2

Type 4:

	a	b	c
a	1	2	2
b	1	1	2
c	1	1	2

Type 5:

	a	b	c
a	1	2	3
b	1	1	2
c	0	1	2

Type 6:

	a	b	c
a	1	2	3
b	1	1	3
c	0	0	2

So if $\gamma_\mu(x, y)$ denotes the entry corresponding to row x and column y in the matrix for Type μ, $\mu = 1, 2, \cdots, 6$, then we have:

(5.3) $\gamma_\mu(a, a) = \gamma_\mu(b, b) = 1$, $\gamma_\mu(c, c) = 2$, $\mu = 1, 2, \cdots, 6$,

extending (5.1). In addition we have the *conservation law*

(5.4) $\gamma_\mu(x, y) + \gamma_\mu(y, x) = 3$, $\mu = 1, 2, \cdots, 6$, $x \neq y$.

Under the standard transformations, the conservation laws (5.3) and (5.4) may be recast in the form given below:

(5.5)   *The average values of the lower bounds for the differences depending on the residue class of e_i (mod 3) add up to ten.*

More precisely, the lower bounds $3, 3, 4$ in (4.7) defining $S_1(n)$ add up to 10. Similarly the lower bounds $3, 2, 5$ of Andrews in (4.8) also add up to 10. For Type 3 partitions counted by $S_3(n)$, the average of 1 and 3 is 2, and so the sum of the averages of the lower bounds in (4.9) is $2 + 2 + 6 = 10$. Similarly in (4.10), with 4 as the average of 3 and 5 we have $2 + 4 + 4 = 10$. Note that in (4.11) with 3 and 5 having average 4, and 4 and 6 having average 5, we have $1 + 4 + 5 = 10$. Finally in (4.11), interpreting the average between 2, 3 and 5 as 3, we have $1 + 6 + 3 = 10$. Thus (5.5) has been verified.

6. Multinomial coefficient expansions

The proof of Lemma 1 we presented in §2 made use of the recurrence (2.9) for the q-multinomial coefficients. The function G in Lemma 1 is the generating function of all Type 1 partitions using i red, j blue and k purple parts. In view of Theorem 6, the generating function will be identical for all Type μ partitions, $\mu = 1, 2, \cdots, 6$, using the same number of parts in the three colors. In this section we give a direct proof of this fact (i.e. without the use of Theorem 6) by means of all six recurrences satisfied by the q-multinomial coefficients. Thus Theorem 6 is a consequence of the results of this section.

The q-multinomial coefficient $\begin{bmatrix} i+j+k \\ i,j,k \end{bmatrix}$ satisfies a total of six basic recurrences depending upon the order in which we reduce each of the letters i, j and k by one. One recurrence is already given in (2.9). Two other recurrences are

(6.1)
$$\begin{bmatrix} i+j+k \\ i,j,k \end{bmatrix} = \begin{bmatrix} i+j+k-1 \\ i-1,j,k \end{bmatrix} + q^i \begin{bmatrix} i+j+k-1 \\ i,j,k-1 \end{bmatrix} + q^{i+k} \begin{bmatrix} i+j+k-1 \\ i,j-1,k \end{bmatrix}$$

and

(6.2)
$$\begin{bmatrix} i+j+k \\ i,j,k \end{bmatrix} = \begin{bmatrix} i+j+k-1 \\ i,j,k-1 \end{bmatrix} + q^k \begin{bmatrix} i+j+k-1 \\ i-1,j,k \end{bmatrix} + q^{k+i} \begin{bmatrix} i+j+k-1 \\ i,j-1,k \end{bmatrix}.$$

There are three more recurrences which we have not written down here.

Next, let $H_a^{(\mu)}(i, j, k)$, $H_b^{(\mu)}(i, j, k)$, and $H_c^{(\mu)}(i, j, k)$ denote the generating function of all minimal Type μ partitions using i red, j blue and k purple parts, such that the smallest part is a_1, b_1 and c_2 respectively. Then these generating functions can be computed by induction on $i + j + k$ in a manner similar to (2.8), as will be described presently.

For Type 1 partitions the starting triple for Scheme 1 ordering is

Scheme 1 : $a_1 \prec b_1 \prec c_2$.

Hence the generating functions $H_a^{(1)}$, $H_b^{(1)}$ and $H_c^{(1)}$ in (2.8) are given in terms of q-multinomial coefficients obtained by replacing i by $i-1$, j by $j-1$ and k by $k-1$, in that order.

For Type 2 partitions the starting triple for Scheme 2 ordering is

$$\text{Scheme 2}: \qquad a_1 \prec c_2 \prec b_1.$$

So, in this case we have

$$H_a^{(2)}(i,j,k) = q^{T_{i+j+k}+T_k} \begin{bmatrix} i+j+k-1 \\ i-1,\, j,\, k \end{bmatrix},$$

(6.3)
$$H_c^{(2)}(i,j,k) = q^{T_{i+j+k}+T_k+i} \begin{bmatrix} i+j+k-1 \\ i,\, j,\, k-1 \end{bmatrix},$$

$$H_b^{(2)}(i,j,k) = q^{T_{i+j+k}+T_k+i+k} \begin{bmatrix} i+j+k-1 \\ i,\, j-1,\, k \end{bmatrix}.$$

Equation (6.3) can be established by induction on $i+j+k$. In Scheme 2, since a_1 occurs first followed by c_2 and then by b_1, we reduced the letters i, j, k by 1 in that order in (6.3). From (6.1) and (6.3) it follows that

(6.4)
$$H_a^{(2)} + H_b^{(2)} + H_c^{(2)} = H = \frac{G}{(q)_{i+j+k}}$$

with H as in (2.5). Thus Scheme 2 corresponds to the recurrence (6.1) just as Scheme 1 corresponds to (2.9).

Similarly, since the starting triple for Scheme 3 ordering is

$$\text{Scheme 3}: \qquad c_2 \prec a_1 \prec b_1,$$

we have

$$H_c^{(3)}(i,j,k) = q^{T_{i+j+k}+T_k} \begin{bmatrix} i+j+k-1 \\ i,\, j,\, k-1 \end{bmatrix},$$

(6.5)
$$H_a^{(3)}(i,j,k) = q^{T_{i+j+k}+T_k+k} \begin{bmatrix} i+j+k-1 \\ i-1,\, j,\, k \end{bmatrix},$$

$$H_b^{(3)}(i,j,k) = q^{T_{i+j+k}+T_k+k+i} \begin{bmatrix} i+j+k-1 \\ i,\, j-1,\, k \end{bmatrix}.$$

This is easily established by induction on $i+j+k$. From (6.5) and (6.2) it follows that

$$H_a^{(3)} + H_b^{(3)} + H_c^{(3)} = H,$$

and so Scheme 3 corresponds to the third recurrence (6.2) for q-multinomial coefficients.

The treatment of the generating functions $H_a^{(\mu)}$, $H_b^{(\mu)}$ and $H_c^{(\mu)}$, for $\mu = 4, 5, 6$ is similar. We record the ideas of this section in the form of

Theorem 10. *The six schemes correspond to the six ways in which the q-multinomial coefficient $\begin{bmatrix} i+j+k \\ i,j,k \end{bmatrix}$ can be expanded in terms of*

$$\begin{bmatrix} i+j+k-1 \\ i-1,\, j,\, k \end{bmatrix}, \qquad \begin{bmatrix} i+j+k-1 \\ i,\, j-1,\, k \end{bmatrix}, \qquad \begin{bmatrix} i+j+k-1 \\ i,\, j,\, k-1 \end{bmatrix}.$$

In particular, under the standard transformations, the six companions functions S_μ, $\mu = 1, 2, \cdots, 6$, to Schur's partition function $S(n)$ correspond to these six recurrences for the q-multinomial coefficients.

7. CONSEQUENCES OF THE BIJECTIONS

In § 4 we constructed bijections $\mathscr{F}_{\mu,\omega}$, $1 \leq \mu, \omega \leq 6$, which converted partitions π_μ of Type μ to partitions π_ω of Type ω. That is,

$$\mathscr{F}_{\mu,\omega}(\pi_\mu) = \pi_\omega$$

and

$$\mathscr{F}_{\mu,\omega}^{-1}(\pi_\omega) = \mathscr{F}_{\omega,\mu}(\pi_\omega) = \pi_\mu.$$

The following are obvious invariants under the mappings $\mathscr{F}_{\mu,\omega}$. For all μ and ω:

$$\sigma(\pi_\mu) = \sigma(\pi_\omega),$$

(7.1)

$$\nu_a(\pi_\mu) = \nu_a(\pi_\omega), \quad \nu_b(\pi_\mu) = \nu_b(\pi_\omega), \quad \nu_c(\pi_\mu) = \nu_c(\pi_\omega).$$

In addition there are other invariants depending on the values μ and ω.

For instance, under the map $\mathscr{F}_{1,2}$ discussed in § 3, the positions occupied by the a-parts do not change. Also the weights of the a-parts remains invariant under $\mathscr{F}_{1,2}$. In view of this, it is convenient to introduce the function $F_\mu(x_m^{(\mu)}; a, b, c)$, which is the generating function for all Type μ partitions whose parts are $\preceq x_m^{(\mu)}$. (Recall that $x_m^{(\mu)}$ denotes the symbol occupying the position m in Scheme μ.) That is,

$$(7.2) \qquad F_\mu(x_m^{(\mu)}) = F_\mu(x_m^{(\mu)}; a, b, c) = \sum_{\substack{\pi_\mu \text{ of Type } \mu \\ \lambda(\pi_\mu) \preceq x_m^{(\mu)}}} a^{\nu_a(\pi_\mu)} b^{\nu_b(\pi_\mu)} c^{\nu_c(\pi_\mu)} q^{\sigma(\pi_\mu)}.$$

Therefore the invariance of the positions and weights of the a-parts under the map $\mathscr{F}_{1,2}$, yields

$$(7.3) \qquad F_1(a_m; a, b, c) = F_2(a_m; a, b, c).$$

Next, given a Type 1 partition π_1 whose parts are $\preceq a_m$, we see that either $\lambda(\pi_1) = a_m$ or $\lambda(\pi_1) \preceq c_m$. If $\lambda(\pi_1) = a_m$, then by deleting this part we get a Type 1 partition with largest part $\preceq a_{m-1}$. Thus from (7.2) we see that

$$(7.4) \qquad F_1(a_m; a, b, c) = a q^{a_m} F_1(a_{m-1}; a, b, c) + F_1(c_m; a, b, c).$$

By means of a similar analysis of Type 2 partitions we have

$$(7.5) \qquad F_2(a_m; a, b, c) = a q^{a_m} F_2(a_{m-1}; a, b, c) + F_2(b_{m-1}; a, b, c).$$

Now (7.3), (7.4) and (7.5) yield

$$(7.6) \qquad F_1(c_m; a, b, c) = F_2(b_{m-1}; a, b, c).$$

There is a nice explanation for the equalities (7.3) and (7.6). The a_m occupy position $3m - 2$ is Scheme 1 and Scheme 2. Also, the set of symbols $\preceq a_m$ in Scheme 1 equals the set of symbols $\preceq a_m$ in Scheme 2. This is why (7.3) holds. Similarly, the c_m in Scheme 1 and b_{m-1} in Scheme 2 occupy position $3m - 3$. Also the set of symbols $\preceq c_m$ in Scheme 1 equals the set of symbols

$\preceq b_{m-1}$ in Scheme 2, explaining the validity of (7.6). At this stage it is worth pointing out that

$$F_1(b_{m-1}; a, b, c) \neq F_2(c_m; a, b, c),$$

even though b_{m-1} in Scheme 1 and c_m in Scheme 2 occupy position $3m - 1$. This is because the set of symbols $\preceq b_{m-1}$ in Scheme 1 is not equal to the set of symbols $\preceq c_m$ in Scheme 2.

Analogous to (7.3) we have

(7.7) $F_1(c_{m+1}; a, b, c) = F_4(c_{m+1}; a, b, c),$

(7.8) $F_2(b_m; a, b, c) = F_3(b_m; a, b, c),$

(7.9) $F_5(a_m; a, b, c) = F_6(a_m; a, b, c).$

To prove (7.7), observe that the positions occupied by the c_m remain invariant under $\mathscr{F}_{1,4}$. Similarly, to get (7.8) note that under the map $\mathscr{F}_{2,3}$, the positions occupied b_m do not change. Finally, (7.9) follows from the invariance of positions occupied by a_m under $\mathscr{F}_{5,6}$. In addition to these, observe also that the positions occupied by c_{m+1}, b_m and a_m in (7.7), (7.8) and (7.9) are all equal to $3m$. Upon comparing (7.6), (7.7) and (7.8) we arrive at

(7.10)
$$F_1(c_{m+1}; a, b, c) = F_2(b_m; a, b, c) = F_3(b_m; a, b, c) = F_4(c_{m+1}; a, b, c).$$

Next, observe that under the mapping $\mathscr{F}_{3,6}$ the positions occupied by c_{m+1} remains invariant. Therefore

(7.11) $F_3(c_{m+1}; a, b, c) = F_6(c_{m+1}; a, b, c).$

In the lexicographic ordering of Schemes 3 and 6, the c_{m+2} occupy position $3m + 1$. The symbols occupying the positions $3m$ in Schemes 3 and 6 are b_m and a_m respectively (see (4.3) and (4.6)). Therefore, in analogy with (7.4) and (7.5) we have

(7.12) $F_3(c_{m+2}; a, b, c) = c\,q^{m+2}F_3(b_{m-1}; a, b, c) + F_3(b_m; a, b, c)$

and

(7.13) $F_6(c_{m+2}; a, b, c) = c\,q^{m+2}F_6(a_{m-1}; a, b, c) + F_6(a_m; a, b, c).$

Hence from (7.11), (7.12) and (7.13), we see by induction on m that

(7.14) $F_3(b_m; a, b, c) = F_6(a_m; a, b, c).$

Finally, combining (7.9), (7.10) and (7.14), we obtain

(7.15)
$$F_1(c_{m+1}; a, b, c) = F_2(b_m; a, b, c) = F_3(b_m; a, b, c)$$
$$= F_4(c_{m+1}; a, b, c) = F_5(a_m; a, b, c) = F_6(a_m; a, b, c).$$

Having established (7.15), we may explain these equalities by observing that the symbols a_m in Schemes 5 and 6, b_m in Schemes 2 and 3, and c_{m+1} in Schemes 1 and 4 all occupy the same position, namely $3m$. In addition, the set of symbols in all six schemes are the same when truncated at position $3m$. That is,

(7.16)
$$\left\{ x_1^{(\mu)}, x_2^{(\mu)}, \cdots, x_{3m}^{(\mu)} \right\}$$
$$= \{ a_1, a_2, \cdots, a_m, \ b_1, \cdots, b_m, \ c_2, \cdots, c_{m+1} \},$$
$$\mu = 1, 2, \cdots, 6.$$

Thus (7.15) may be recast in the following more elegant form.

Theorem 11. *With* $F_\mu(x_m^{(\mu)})$ *as in (7.2) we have*

$$F_1(x_{3m}^{(1)}) = F_2(x_{3m}^{(2)}) = \cdots = F_6(x_{3m}^{(6)}).$$

It is to be noted that the proof of Theorem 11 and (7.15) given above makes use of the bijections $\mathscr{F}_{\mu,\omega}$ and recurrences like (7.3), and is not via (7.16). It might be worthwhile to give a purely algebraic proof of Theorem 11 based on (7.16).

Under the standard transformations something quite striking happens in (7.16), namely, all the sets are equal to

$$\{1, 2, 3, \cdots, 3m\}.$$

That is, the set of symbols $\preceq x_{3m}^{(\mu)}$ in Scheme μ is equal to the set of positive integers $\leq 3m$. Thus, if $K_\mu(m; a, b, c) = K_\mu(m)$ represents the a-b-c-q generating function of all partitions of Type μ with the additional restriction that all parts are $\leq m$, then we have

Theorem 12. $K_1(3m) = K_2(3m) = \cdots = K_6(3m)$.

Theorem 2 is a corollary of Theorem 12 by setting $a = b = c = 1$ and comparing the coefficients of q^n. In addition, Theorem 8 is a consequence of Theorem 12 by letting $n \to \infty$. Thus Theorems 11 and 12 yield substantial generalizations and refinements of Theorems A and S.

8. Concluding remarks

The method of weighted words provided the general setting from which Schur's theorem, the companion results, the refinements involving parameters a, b and c, and generalizations to an arbitrary modulus $M \geq 3$ all emerged. We have been able to extend the method and apply it to other partition problems as well.

For Schur's theorem we started out with two primary colors a (red) and b (blue), and one secondary color $c = ab$ (purple). In a subsequent paper written jointly with Andrews [2], we view deep theorems of Göllnitz ([13], Sätze 4.8 and 4.10) as emerging out of a situation where we have three primary colors a, b and c, and three secondary colors ab, ac and bc. Indeed, Schur's theorem corresponds to the special case obtained by setting $c = 0$, in which case c, ac and bc drop out and only a, b and ab survive.

Motivated by a study of Lie algebras, Capparelli [12] recently conjectured a partition theorem for which Andrews [9] gave a proof using generating functions. In a joint paper written with Andrews [3], we obtain refinements and generalizations of Capparelli's theorem by the method of weighted words involving free parameters a, b and c. The bijections discussed in §3 can be modified to apply to the Capparelli problem.

it was the effort to understand the relationship between Theorem A and Theorem S that led to this paper.

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